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EXACT SOLUTIONS OF THE FOKKER-PLANCK-KOLMOGOROV EQUATION FOR CERTAIN MULTIDIMENSIONAL DYNAMIC SYSTEMS*

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Analytic solutions of the Fokker-Planck-Kolmogorov equations for the stationary joint probability densities of the state variables are obtained for one class of multidimensional nonlinear dynamic systems with external random perturbations of white-noise type, and for one class of multidimensional linear dynamic systems with simultaneously acting external and parametric random perturbations of white-noise type. The behaviour of the Lotki-Volterra system in a random medium is investigated as an example.

1. Consider the system of stochastic differential equations

$$x_i^{\cdot} = \frac{\partial H}{\partial y_i}, \quad y_i^{\cdot} = -\frac{\partial H}{\partial x_i} - g(H)\frac{\partial H}{\partial y_i} + \zeta_i(t) \tag{1.1}$$
$$H = H(x_1, \ldots, x_n;\ y_1, \ldots, y_n), \ i = 1, \ldots, n$$

Here $\zeta_i(t)$ are independent stationary normal centred random processes of white-noise type of similar intensity D: $\langle \zeta_i(t)\zeta_j(t+\tau)\rangle = D\delta_{ij}\delta(\tau)$, where δ_{ij} is the Kronecker delta, $\delta(\tau)$ is the delta function, and the angle brackets denote averaging. The joint probability density $p(x_1, \ldots, x_n;\ y_1, \ldots, y_n;\ t)$ of the variables $x_i(t), y_i(t)$ satisfies the Fokker-Planck-Kolmogorov equation /1,2/

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial y_i}p\right) + \sum_{i=1}^{n}\frac{\partial}{\partial y_i}\left[\frac{\partial H}{\partial x_i}p + g(H)\frac{\partial H}{\partial y_i}p\right] + \frac{D}{2}\sum_{i=1}^{n}\frac{\partial^2 p}{\partial y_i^2} \tag{1.2}$$

By direct substitution it can be shown that Eq.(1.2) has the following stationary $(\partial p/\partial t \equiv 0)$ solution

$$p(x_1, \ldots, x_n;\ y_1, \ldots, y_n) = C\exp[-(2/D)G(H)] \tag{1.3}$$
$$G(H) = \int_0^H g(H')\,dH'$$

Here C is a constant to be determined from the normalization condition (it is clear that solution (1.3) will in fact determine the desired stationary probability density only if a normalization integral exists). For the special case

$$H(x_1, \ldots, x_n;\ y_1, \ldots, y_n) = V(x_1, \ldots, x_n) + \frac{1}{2}\sum_{i=1}^{n} y_i^2$$

distribution (1.3) was obtained in /3/.

As an example let us consider a modified Lotki-Volterra system describing the fluctuation in the sizes of two interacting populations of the "predator-prey" type in a random medium /4,5/

$$u^{\cdot} = k\beta uv - mu, \quad v^{\cdot} = \alpha v\,[1 + \xi(t)] - \beta uv - \gamma v^2 \tag{1.4}$$

Here $u(t)$ and $v(t)$ are the sizes of the two populations; $k, \beta, m, \alpha, \gamma$ are positive constants, and $\xi(t)$ is a stationary normal centred random process of white-noise type with intensity D_ξ. In /4/ the problem being examined was investigated by analyzing the stochastic mean-square stability of the equation for small perturbations, obtained by linearizing the Eqs.(1.4) in a neighbourhood of a stable equilibrium position

$$u_0 = \alpha/\beta - \gamma m/(k\beta^2), \quad v_0 = m/(k\beta) \tag{1.5}$$

(Here and henceforth we assume that $\gamma < \alpha k\beta/m$).

By the change of variables

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$$x = \ln u, \quad y = \ln v \tag{1.6}$$

Eqs. (1.4) can be reduced to the form (1.1), and (since the index i takes only the one value $n = 1$, we will omit it)

$$H(x, y) = k\beta e^y - my + \beta e^x - (\alpha - \gamma m/(k\beta))\, x \tag{1.7}$$
$$g(H) = \gamma / (k\beta), \ D = D_\xi \alpha^2$$

Substituting (1.7) into (1.3) and returning to the original variables u and v, by the well-known rules for finding the probability density of a function of a random variable /1,6/, using (1.5) we obtain, after normalizing in the first quadrant of the plane u, v the following expression for the stationary joint probability density $w(u, v)$ of the two population sizes (Γ is the gamma function)

$$w(u, v) = w_1(u)\, w_2(v), \quad w_1(u) = (\delta/k)^{\delta u_0/k} \Gamma^{-1}(\delta u_0/k)\, u^{\delta u_0/k - 1} e^{-\delta u/k} \tag{1.8}$$
$$w_2(v) = \delta^{\delta v_0} \Gamma^{-1}(\delta v_0)\, v^{\delta v_0 - 1} e^{-\delta v}, \quad \delta = 2\gamma/(D_\xi \alpha^2)$$

Thus, for steady-state fluctuations of system (1.4) the processes $u(t)$ and $v(t)$ are statistically independent and are subject to one and the same distribution law χ^2. The means of processes $u(t)$, $v(t)$ equal u_0, v_0, respectively, the variances equal $u_0 k/\delta$, v_0/δ, respectively, and the largest probable values (the maximum points of the functions $w_1(u)$, $w_2(v)$) equal $u_0 - k/\delta$, $v_0 - 1/\delta$, respectively. $D_\xi \to 0$ the probability densities $w_1(u)$ and $w_2(v)$ are asymptotically normal, while for fairly intense random perturbations — when $\delta < k/u_0$ and $\delta < 1/v_0$ respectively — they are monotonically decreasing on the semi-axes $u > 0$, $v > 0$ and have singularities at the points $u = 0$, $v = 0$. Such a qualitative transformation of the function $w(u, v)$ in the domain of large D_ξ does not at all signify, however, the death of the populations, since the singularities mentioned are integrable, i.e., for any positive u_0, v_0, δ expression (1.8) does indeed represent the joint stationary probability density of the processes $u(t)$ and $v(t)$. It is clear that the deduction that the populations do not vanish, valid only within the framework of the present model, does not take into account those effects which are connected with the discreteness of the real processes $u(t)$ and $v(t)$ and which may become essential when the values of u, and v are not sufficiently large compared with unity.

Let us find the average per unit time of the number $n_+(u)$ of intersections by the process $u(t)$ of some level u with a positive derivative $u^{\cdot} = z$. Let $p(u, z)$ be the stationary joint probability density of the processes $u(t)$ and $z(t)$. Making use of a well-known /1/ expression for $n_+(u)$ and expressing z in terms of u, v in accordance with the first equation in (1.4), we have

$$n_+(u) = \int_0^\infty z p(u, z)\, dz = \int_{v_0}^\infty k\beta u (v - v_0)\, w(u, v)\, dv \tag{1.9}$$

Substituting (1.8) into (1.9) and carrying out the integration, we obtain

$$n_+(u) = \frac{(k\beta/\delta)\,(\delta v_0)^{\delta v_0}\,(\delta u/k)^{\delta u_0/k} \exp(-\delta v_0 - \delta u/k)}{\Gamma(\delta v_0)\,\Gamma(\delta u_0/k)} \tag{1.10}$$

In particular, from (1.10) we obtain (on the basis of the asymptotic representation of the gamma function in the domain of large values of the argument /7/) the formula

$$\lim_{\delta \to \infty} n_+(u_0) = \frac{\Omega}{2\pi}, \quad \Omega = \left(\alpha m - \frac{\gamma m^2}{k\beta}\right)^{1/2}$$

The quantity Ω is the natural frequency of small fluctuations of system (1.4) in the neighbourhood of the stable equilibrium position u_0, v_0.

2. We consider the system of stochastic differential equations

$$x_i^{\cdot} = -\beta x_i\,[1 + \xi(t)] + \zeta_i(t); \quad i = 1, \ldots, n \tag{2.1}$$

to be understood in Stratonovich's sense. Here $\xi(t)$, $\zeta_i(t)$ are independent stationary centred normal random processes of white-noise type, and $\langle \xi(t)\, \xi(t+\tau)\rangle = D_\xi \delta(\tau)$, $\langle \zeta_i(t)\, \zeta_i(t+\tau)\rangle = D_\zeta \delta(\tau)$, $i = 1, \ldots, n$. The Fokker–Planck – Kolmogorov equation for the joint probability density $p(x_1, \ldots, x_n; t)$ can be written, according to /2/, as

$$\frac{\partial p}{\partial t} = \beta \sum_{i=1}^n \frac{\partial}{\partial x_i}(x_i p) + \frac{D_\zeta}{2} \sum_{i=1}^n \frac{\partial^2 p}{\partial x_i^2} + \frac{D_\xi \beta^2}{2} \sum_{i=1}^n \frac{\partial}{\partial x_i}\left[x_i \sum_{j=1}^n \frac{\partial}{\partial x_j}(x_j p)\right] \tag{2.2}$$

Equation (2.2) has the stationary ($\partial p/\partial t \equiv 0$) solution

$$p(x_1, \ldots, x_n) = C\left(\varkappa + \sum_{i=1}^n x_i^2\right)^{-\delta}, \qquad \varkappa = D_\zeta/D_\xi, \quad \delta = 1/\beta D_\xi + n/2 \tag{2.3}$$

(as before, C is a normalizing constant). Solution (2.3) determines the stationary joint probability density of the variables $x_i(t)$ when the normability condition $2\delta > 2/(\beta D_\xi) + n$ is satisfied, i.e., when $\beta > 0$. We see that this condition is identical with the condition of stochastic probability-stability of system (2.1) with $\zeta_i(t) \equiv 0$ /8/.

We consider further the following system of Stratonovich stochastic differential equations

$$A_i^{\cdot} = -\alpha A_i + 2D_\xi A_i + D_\zeta/A_i + (2D_\zeta)^{1/2}\zeta_i'(t) - (2D_\xi)^{1/2}A_i\xi'(t); \quad i = 1, \ldots, n \tag{2.4}$$

Here $\xi'(t)$, $\zeta_i'(t)$ are independent stationary centered normal random processes of white-noise type of unit intensity. Changing in (2.4) to the new variables $V_i = A_i^2$, we can set up the following Fokker–Planck–Kolmogorov equation for the joint probability density $p(V_1, \ldots, V_n; t)$ of the variables $V_i(t)$

$$\frac{\partial p}{\partial t} = \sum_{i=1}^{n} \frac{\partial}{\partial V_i}[(2\alpha - 4D_\xi)V_i p] - 2D_\zeta \sum_{i=1}^{n} \frac{\partial p}{\partial V_i} + \tag{2.5}$$

$$4D_\zeta \sum_{i=1}^{n} \frac{\partial}{\partial V_i}\left[\sqrt{V_i}\frac{\partial}{\partial V_i}\left(\sqrt{V_i}\,p\right)\right] + 4D_\xi \sum_{i=1}^{n} \frac{\partial}{\partial V_i}\left[V_i \sum_{j=1}^{n} \frac{\partial}{\partial V_j}(V_j p)\right]$$

The stationary solution of Eq.(2.5) is

$$p(V_1, \ldots, V_n) = C\left(\varkappa + \sum_{i=1}^{n} V_i\right)^{-\delta} \tag{2.6}$$

$$\varkappa = D_\zeta/D_\xi, \quad \delta = \alpha/2D_\xi + n - 1$$

and really represents the joint stationary probability density of $V_i(t)$ when the normability condition $\delta > n$ is satisfied, i.e., when $\alpha/2D_\xi > 1$; this condition is identical with the probability-stability condition of system (2.4) with $\zeta_i'(t) \equiv 0$.

Solution (2.6) approximately determines the joint stationary probability density of the squares of the amplitudes of the mixing of identical unconnected oscillators with a common random parametric perturbation, located in a field of random external forces. Let the equations of motion of the osicllators be

$$x_i^{\cdot\cdot} + 2\alpha x_i^{\cdot} + \Omega^2 x_i[1 + \xi(t)] = \zeta_i(t); \quad i = 1, \ldots, n \tag{2.7}$$

where $\xi(t)$, $\zeta_i(t)$ are broadband stationary centred random processes with spectral densities $\Phi_{\xi\xi}(\omega)$, $\Phi_{\zeta_i\zeta_j}(\omega) = \Phi_{\zeta\zeta}(\omega)\delta_{ij}$, and the quantities α, Φ are small. Then in (2.7) we can change to the new variables $A_i(t)$, $\varphi_i(t)$ as given by the relations

$$x_i = A_i\cos\theta_i, \quad x_i^{\cdot} = -\Omega A_i\sin\theta_i, \quad \theta_i = \Omega t + \varphi_i$$

A subsequent application of the theorem in /9/ leads to a system of Ito stochastic equations in $A_i(t)$ (see /6/ for one such equation), and this system proves to be exactly equivalent to the system of Stratonovich equations (2.4) when

$$D_\zeta = {}^1/_2\,\pi\Omega^{-2}\Phi_{\zeta\zeta}(\Omega), \quad D_\xi = {}^1/_8\,\pi\Omega^2\Phi_{\xi\xi}(2\Omega)$$

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